A rationality result for Kovacic's algorithm

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1 Statement of the result

Consider the second order linear differential equation $y'' = ry$, with $r \in \mathbf{Q}(x)$. Let $\mathbf{Q}^{cl}$ denote the algebraic closure of the field of rational numbers $\mathbf{Q}$ and let G denote the differential Galois group over $\mathbf{Q}^{cl}(x)$ of this equation. Then $G \subset Sl(2, \mathbf{Q}^{cl})$. For any solution $y \neq 0$ of the equation the element $u = \frac{y'}{y}$ satisfies the Riccati equation $u' + u^2 = r$. The main result in [Kov86] is the following theorem.

Theorem 1.1 *(See [Kov86])*

1. *If there is no algebraic solution over $\mathbf{Q}^{cl}(x)$ of the Riccati equation then $G = Sl(2, \mathbf{Q}^{cl})$.*
2. *If there is an algebraic solution of the Riccati equation then the minimal degree n of such an equation can be $1, 2, 4, 6, 12$ and*
   (a) *if $n = 1$ then $G \subset \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \mid c \in \mathbf{Q}^{cl} \setminus \{0\}, d \in \mathbf{Q}^{cl} \right\}$.*
   (b) *if $n = 2$ then $G = D_\infty$ or $G = D_m$ with $m \geq 3$.*
   (c) *if $n = 4$ then G is the tetrahedral group.*
   (d) *if $n = 6$ then G is the octahedral group.*
   (e) *if $n = 12$ then G is the icosahedral group.*

The list of conjugacy classes of algebraic subgroups of $Sl(2, \mathbf{Q}^{cl})$ appearing in above theorem is well known to be exhaustive. The Kovacic's algorithm for the calculation of an algebraic solution u uses algebraic extensions of $\mathbf{Q}$ of arbitrary degree. In [B92] a "rational" version of the Kovacic algorithm is indicated for $n = 2$.

In this paper we want to proof the following rationality result.

Theorem 1.2 *Suppose that the Riccati equation $u' + u^2 = r$ has a solution, which is algebraic over $\mathbf{Q}^{cl}(x)$. Then there exists an algebraic solution u of minimal degree n of the Riccati equation such that the coefficients of the minimum polynomial of u over $\mathbf{Q}^{cl}(x)$ lie in a field $K(x)$ with $[K : \mathbf{Q}] \leq 2$. Moreover, only in the cases: $n = 1$ and G is the multiplicative group $\mathbf{G}_m$ or a finite cyclic group of order > 2 or $n = 4$ and G the tetrahedral group, a field extension K of degree 2 of $\mathbf{Q}$ can be needed.*

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2 The proof

Again let $r \in \mathbf{Q}(x)$ and consider the differential equation $y'' = ry$. Suppose that $\alpha \in \mathbf{Q}$ is a regular point of the differential equation. Then there exists two independent solutions $y_0, y_1 \in \mathbf{Q}[[x-\alpha]]$ of this equation. Let $F = \mathbf{Q}^{cl}(x, y_0, y_1, y_0', y_1')$. The differential field $F \subset \mathbf{Q}^{cl}((x-\alpha))$ is a Picard-Vessiot extension of $\mathbf{Q}^{cl}(x)$ associated with the equation $y'' = ry$. By $DGal(E/F)$ of an extension of differential fields we denote the group of F-linear automorphisms of the field E commuting with the differentiation.

Lemma 2.1 *The sequence*

$$1 \to DGal(F/\mathbf{Q}^{cl}(x)) \xrightarrow{i} DGal(F/\mathbf{Q}(x))$$

$$\xrightarrow{\rho} Gal(\mathbf{Q}^{cl}/\mathbf{Q}) \to 1$$

is split exact.

Proof. We give a description of a splitting homomorphism $\tilde{s} : Gal(\mathbf{Q}^{cl}/\mathbf{Q}) \to DGal(F/\mathbf{Q}(x))$. Let $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ and $\sum_{i=k}^{\infty} a_i(x-\alpha)^i \in \mathbf{Q}^{cl}((x-\alpha))$. Then we define the group homomorphism

$$s : Gal(\mathbf{Q}^{cl}/\mathbf{Q}) \to DGal(\mathbf{Q}^{cl}((x-\alpha))/\mathbf{Q}(x))$$

$$\text{by } (s\sigma).\sum_{i=k}^{\infty} a_i(x-\alpha)^i = \sum_{i=k}^{\infty} \sigma(a_i)(x-\alpha)^i$$

It is easily seen that $s\sigma$ is a $\mathbf{Q}(x)$-linear automorphism of $\mathbf{Q}^{cl}((x-\alpha))$ and that $s\sigma$ and $\frac{d}{dx}$ commute. Further for all $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ we have $(s\sigma)(F) = F$, because $(s\sigma)\mathbf{Q}^{cl}(x) = \mathbf{Q}^{cl}(x)$ and $(s\sigma)(y_i^{(j)}) = y_i^{(j)}$ for $i, j = 0, 1$. We define $\tilde{s}\sigma$ to be the restriction of $s\sigma$ to F. It is clear that $\rho \circ \tilde{s} = id$. Hence $\tilde{s} : Gal(\mathbf{Q}^{cl}/\mathbf{Q}) \to$

$DGal(F/\mathbf{Q}(x))$ is a splitting homomorphism. The exactness of the sequence is now obvious.

Proof of theorem 1.2. The proof is given case by case.

1. Suppose $n = 1$. There are three possibilities.
   (a) If G contains the additive group $\mathbf{G}_a$ then the Riccati equation has only one solution $u \in \mathbf{Q}^{cl}(x)$. If $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)u$ is also a solution of the Riccati equation. Hence $(\tilde{s}\sigma)u = u$ and $u \in \mathbf{Q}(x)$.
   (b) If G is the multiplicative group $\mathbf{G}_m$ or a finite cyclic group of order greater than 2 then there are exactly two solutions $u_1, u_2 \in \mathbf{Q}^{cl}(x)$. Hence for all $\sigma \in Gal(\mathbf{Q}^{cl/}/\mathbf{Q})$ we have $(\tilde{s}\sigma)\{u_1, u_2\} = \{u_1, u_2\}$. We conclude that $u_1, u_2 \in K(x)$, where K is a field such that $[K : \mathbf{Q}] \leq 2$.
   (c) If G is a cyclic group of order equal to 1 or 2 then the Riccati equation has infinitely many solutions $u \in \mathbf{Q}^{cl}(x)$. Let $u_0 = \frac{y_0'}{y_0}$. Then $u_0 \in \mathbf{Q}^{cl}(x) \cap \mathbf{Q}((x - \alpha)) = \mathbf{Q}(x)$ and u_0 satisfies the Riccati equation.

2. Suppose $n = 2$. Now $G = D_\infty$ or $G = D_m$ with $m \geq 3$. It is not difficult to verify that there are exactly two solutions u_1 and u_2 of the Riccati equation, which are quadratic over $\mathbf{Q}^{cl}(x)$. These solutions have the same minimum polynomial over $\mathbf{Q}^{cl}(x)$. If $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)\{u_1, u_2\} = \{u_1, u_2\}$. Hence the coefficients of the minimum polynomial over $\mathbf{Q}^{cl}(x)$ are fixed under $\tilde{s}\sigma$ and so the coefficients are already in $\mathbf{Q}(x)$.

3. Suppose $n = 4$. Now G is the tetrahedral group and $\#G = 24$. If u is a solution of the Riccati equation then $H_u = \{\sigma \in G \mid \sigma(u) = u\}$ is a cyclic subgroup of G and u is algebraic of degree $[G : H_u]$ over $\mathbf{Q}^{cl}(x)$. Conversely if $H \subset G$ is a cyclic subgroup then there are solutions u of the Riccati equation algebraic of degree $[G : H]$ over $\mathbf{Q}^{cl}(x)$, which are fixed under the action of H. Moreover, if $\#H \geq 3$ then there are exactly two solutions of the Riccatti equation algebraic of degree $[G : H]$ over $\mathbf{Q}^{cl}(x)$, which are fixed under the action of H. In the tetrahedral case $n = 4$ and there are four cyclic subgroups of G of order 6. Hence there are eight solutions $u_1, \ldots, u_8$ of the Riccati equation which are algebraic of degree 4 over $\mathbf{Q}^{cl}(x)$ and there are two monic irreducible polynomials P, Q of degree 4 over $\mathbf{Q}^{cl}(x)$, such that $(PQ)(u_i) = 0$ for $i = 1, \ldots, 8$. Suppose $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)\{u_1, \ldots, u_8\} = \{u_1, \ldots, u_8\}$ and thus $(\tilde{s}\sigma)(PQ) = PQ$ and $(\tilde{s})\sigma\{P, Q\} = \{P, Q\}$. Hence the coefficients of P and Q lie in $K(x)$ where K satisfies $[K : \mathbf{Q}] \leq 2$.

4. Suppose $n = 6$. Now G is the octahedral group and $\#G = 48$. There are three cylic subgroups of order 8. Hence there are six solutions $u_1, \ldots, u_6$ of the Riccati equation which are algebraic of degree 6 over $\mathbf{Q}^{cl}(x)$ and there is a unique monic irreducible polynomial P such that $P(u_i) = 0$ for $i = 1, \ldots, 6$. Suppose $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)\{u_1, \ldots, u_6\} = \{u_1, \ldots, u_6\}$. Therefore the coefficients of P are in $\mathbf{Q}(x)$.

5. Suppose $n = 12$. Now G is the icosahedral group and $\#G = 120$. There are six cylic subgroups of order 10. Hence there are twelve solutions $u_1, \ldots, u_{12}$ of the Riccati equation which are algebraic of degree 12 over $\mathbf{Q}^{cl}(x)$ and there is a unique monic irreducible polynomial P such that $P(u_i) = 0$ for $i = 1, \ldots, 12$. Suppose $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ then $(\tilde{s}\sigma)\{u_1, \ldots, u_{12}\} = \{u_1, \ldots, u_{12}\}$. It follows that the coefficients of P are in $\mathbf{Q}(x)$.

In section 4 we will see that in the tetrahedral case quadratic field extensions of $\mathbf{Q}$ can occur. However in the tetrahedral case there are exactly 6 solutions of the Riccati equation of degree 6. These solutions have the same minimum polynomial P over $\mathbf{Q}^{cl}$. The coefficients of this polynomial are in $\mathbf{Q}(x)$.

3 Remarks on lemma (2.1)

Let V denote the two-dimensional vector space over $\mathbf{Q}^{cl}$ of the solutions of $y'' = ry$ in F. The two-dimensional space $V_0 := \mathbf{Q}y_0 + \mathbf{Q}y_1$ has the property that $V_0 \otimes_{\mathbf{Q}} \mathbf{Q}^{cl} = V$ and the natural action of $Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ on this tensor product coincides with the action on V by the splitting homomorphism $\tilde{s}$.

For simplicity we have made the choice of Q and Q(x) in Thm 1.2. This choice can be replaced (without any change in the proof) by a field C of constants of characteristic zero and the function field $C(X)$ of an absolutely irreducible curve X over C such that X has a C-rational point x_0 which is a regular point of the differential equation. Indeed, the Picard-Vessiot field of the differential equation can be found inside the field of fractions of the completion of the local ring of $X \otimes C^{cl}$ at x_0 and the Galois group of C^{cl}/C acts in a natural way on this field.

In the general case: K a differential field with a (non algebraically closed) field of constants C one denotes by $K^+ = KC^{cl}$ the compositum of the fields K and C^{cl}. The Picard-Vessiot field F of a differential equation over K is seen as a field extension of K^+. One can show (using an algebraic construction of the Picard-Vessiot field) that the following natural sequence is exact:

$$1 \to DGal(F/K^+) \to DGal(F/K)$$

$$\to Gal(K^+/K) \to 1$$

We note that in general $Gal(K^+/K)$ is a proper subgroup of $Gal(C^{cl}/C)$ since K and C^{cl} need not be linearly disjunct over C. We do not know in this general situation that the sequence above splits. However, Thm 1.2 remains valid in this case and takes the form:

Let $n \in \{1, 2, 4, 6, 12\}$ denote the integer corresponding to the differential Galois group $G = DGal(F/K^+)$. There is a subfield L, $K \subset L \subset K^+$ with $[L : K] \leq 2$ and a solution u of the Riccati equation which is algebraic over K^+ with degree n such that the monic minimal equation of u over K^+ has its coefficients in L.

In stead of using the splitting, one can lift any element of $Gal(K^+/K)$ to a differential automorphism of F over K. That suffices for the proof of the various cases, except for the case $n = 1$ and G is a group of order 1 or 2.

There one needs the following ad hoc arguments: If $G = 1$ then $F = K^+$. Let $y \neq 0$ be a solution of the differential equation. Then $K(y)$ is an extension of K of some degree d and equals $K(z)$ where z is a constant. Write $y = f_0 + f_1 z + \ldots + f_{d-1} z^{d-1}$ with all $f_i \in K$. Clearly $f_i'' = rf_i$ and since some $f_i \neq 0$ the Riccati equation has a solution in K.

If G has order 2 then by using an algebraic construction for the Picard-Vessiot field one can show the existence of L as in the statement with $[L : K] \leq 2$.

4 Examples of the quadratic extension

We will show that for $n = 1$ and G the multiplicative group or a finite cyclic group of order > 2 and $n = 4$ and G is the tetrahedral group any quadratic extension K of $\mathbf{Q}$ does occur.

1. G is the multiplicative group $\mathbf{G_m}$ or a finite cyclic group of order > 2. Let the field K be given as $K = \mathbf{Q}(\lambda)$ where $\lambda^2 \in \mathbf{Z}$ is a square free integer $(\neq 0, 1)$. Take $u_0, u_1 \in \mathbf{Q}(x), u_1 \neq 0$ and write $u = u_0 + \lambda u_1$. The equation $u' + u^2 = r \in \mathbf{Q}(x)$ is equivalent to $u_0 = -1/2\frac{u_1'}{u_1}$ and $r = u_0^2 + u_0' + \lambda^2 u_1^2$. Any choice of $u_1 \neq 0$ determines some u_0 and r and an equation $y'' = ry$. The corresponding Riccati equation has at least the two solutions $u_0 \pm \lambda u_1$. The equation $y' = (-1/2\frac{u_1'}{u_1} + \lambda u_1)y$ determines the differential Galois group which can be a finite cyclic group or the multiplicative group $\mathbf{G}_m$.
   For a general choice of u_1 the differential Galois group will be $\mathbf{G}_m$.
   If one chooses $u_1 = \frac{a}{b(x^2-\lambda^2)}$ with $a, b \in \mathbf{Z} \setminus \{0\}$, $b \geq 1$ and $g.c.d(a, b) = 1$ then $y = (x - \lambda)^{(\frac{1}{2} + \frac{a}{2b})}(x + \lambda)^{(\frac{1}{2} - \frac{a}{2b})}$ satisfies the equation $y' = (-1/2\frac{u_1'}{u_1} + \lambda u_1)y$ and G is a finite cyclic group of order b if a is odd and b is odd and of order $2b$ if a is even or b is even.

2. Let L_R denote the differential operator $(\frac{d}{dx})^2 - R$, where $R = \frac{3}{16x(x-1)} - \frac{3}{16x^2} - \frac{2}{9(x-1)^2}$. The differential Galois group of L_R is the tetrahedral group. There are exactly eight solutions $U_1, \ldots, U_8$ of the Riccati equation $U' + U^2 = R$ which are algebraic of degree 4 over $\mathbf{Q}^{cl}(x)$ and there are two monic irreducible polynomials $P_R, Q_R \in \mathbf{Q}^{cl}(x)[T]$ of degree 4 such that $(P_R Q_R)(U_i) = 0$ for $i = 1, \ldots, 8$. One can show that these polynomials satisfy $P_R, Q_R \in \mathbf{Q}(x)[T]$ and $P_R(x, T) = Q_R(\frac{x}{x-1}, T)$. An explicit calculation of P_R, Q_R is done in [Kov86], section 5.2.

   According to F.Klein, the differential operator L_R with the tetrahedral group as differential Galois group is the 'universal' in the following sense:
   Let L_r denote the differential operator $(\frac{d}{dt})^2 - r$ with $r \in \mathbf{Q}^{cl}(t)$ and suppose that the differential Galois group of this equation is also the tetrahedral group. Then there exists exactly two $\mathbf{Q}^{cl}$-linear field endomorphisms $\phi_1, \phi_2 : \mathbf{Q}^{cl}(x) \to \mathbf{Q}^{cl}(t)$ such that $(\phi_i)_* L_R = L_r$. Moreover $\phi_1 = \phi_2 \circ \theta$ where θ is the $\mathbf{Q}^{cl}$-linear field automorphism of $\mathbf{Q}^{cl}(x)$ given by $x \mapsto \frac{x}{x-1}$. The explicit expression for r is

   $$r = R(\phi(x))(\phi'(x))^2 - \frac{1}{2}(\frac{\phi''(x)}{\phi'(x)})' + \frac{1}{4}(\frac{\phi''(x)}{\phi'(x)})^2$$

   with $\phi = \phi_1$ or $\phi = \phi_2$ and $' = \frac{d}{dt}$.
   We refer to [BD79], theorems 3.4 and 3.7, for the statement above and we note that **the 'uniqueness of the pullback' claimed in theorem 3.7 does not hold in the tetrahedral case because $\theta_* L_R = L_R$. In the other two cases (with octahedral or icosahedral group) the uniqueness of the pullback is valid.**

Let ϕ be ϕ_1 or ϕ_2. Let $u_1, \ldots, u_8$ be the eight solutions of the Riccati equation $u' + u^2 = r$,which are algebraic of degree 4 over $\mathbf{Q}^{cl}(t)$. If $P_r = P_R(\phi(x), T)$ and $Q_r = Q_R(\phi(x), T)$ then P_r, Q_r are the unique monic irreducible polynomials such that $(P_r Q_r)(u_i) = 0$ for $i = 1, \ldots, 8$.

We come now to the construction of the example. Fix a field $K = \mathbf{Q}(\lambda)$, where $\lambda^2 \in \mathbf{Z}$ is a square free integer ($\neq 0, 1$). Let $\phi : \mathbf{Q}^{cl}(x) \to \mathbf{Q}^{cl}(t)$ be the $\mathbf{Q}^{cl}$-linear field isomorphism given by $x \mapsto \frac{2}{1+2\lambda t}$. We note that $(\phi \circ \theta)(x) = \frac{2}{1-2\lambda t} = \tau(\phi(x))$, where τ is any automorphism of $\mathbf{Q}^{cl}(t)$ satisfying $\tau(t) = t, \tau(\lambda) = -\lambda$. Then $L := \phi_* L_R$ has the form $(\frac{d}{dx})^2 - r_\lambda$ with

$$r = -\frac{32\lambda^2}{9(1-4\lambda^2 t^2)^2} + \frac{3\lambda^2}{4(1-4\lambda^2 t^2)} \in \mathbf{Q}(t)$$

. The differential Galois group of L is of course the tetrahedral group. Let $P := P_R(\phi(x), T)$ and $Q := Q_R(\phi(x), T)$. Then P and Q are the minimum polynomials of the eight solutions of the Riccati equation $u' + u^2 = r$ which are algebraic of degree 4 over $\mathbf{Q}^{cl}(t)$. Clearly $P, Q \in K(x)[T]$. Take a τ as above and extend τ as an automorphism of $\mathbf{Q}^{cl}(t)[T]$ by $\tau(T) = T$. Then

$$\begin{aligned} \tau P = P_R(\tau(\phi(x)), T) &= P_R(\frac{\phi(x)}{\phi(x) - 1}, T) \\ &= Q_R(\phi(x), T) = Q \end{aligned}$$

Hence P, Q do not belong to $\mathbf{Q}(t)[T]$. This finishes the example.

5 Differential equations of order 3

Consider the third order linear differential equation $y''' + py' + qy = 0$, with $p, q \in \mathbf{Q}(x)$. Let $G \subset Sl(3, \mathbf{Q}^{cl})$ be the differential galois group over $\mathbf{Q}^{cl}(x)$ of this equation. For any solution $y \neq 0$ of the equation the element $u = \frac{y'}{y}$ satisfies the Riccati equation $u'' + 3uu' + u^3 + pu + q = 0$. The analogue of theorem 1.1 can be found in [SU92]. We will use their terminology and description of finite primitive groups.

Definition 5.1 *A group $H \subset Sl(3, \mathbf{Q}^{cl})$ is called 1-reducible if the elements of the group have a common eigenvector.*

Theorem 5.2 *(See [SU92].)*

1. *If there is no algebraic solution over $\mathbf{Q}^{cl}(x)$ of the Riccati equation then $G = Sl(3, \mathbf{Q}^{cl})$ or $G/Z(G) = PSl(2, \mathbf{Q}^{cl})$ or G is a reducible but not a 1-reducible group.*
2. *If there is an algebraic solution of the Riccati equation then the minimal degree n of such an equation can be $1, 3, 6, 9, 21, 36$ and*
   - *(a) if $n = 1$ then G is a 1-reducible group.*
   - *(b) if $n = 3$ then G is an imprimitive group.*
   - *(c) if $n = 6$ then $G/Z(G)$ is isomorphic to F_{36} or A_5.*
   - *(d) if $n = 9$ then $G/Z(G)$ is isomorphic to H_{72} or H_{216}.*
   - *(e) if $n = 21$ then $G/Z(G) = G_{168}$*
   - *(f) if $n = 36$ then $G/Z(G) = A_6$*

Theorem 5.3 *Suppose that the Riccati equation $u'' + 3uu' + u^3 + pu + q = 0$ has a solution, which is algebraic over $\mathbf{Q}^{cl}(x)$. Then there exists an algebraic solution u of minimal degree n of the Riccati equation such that the coefficients of the minimum monic polynomial of u over $\mathbf{Q}^{cl}(x)$ lie in a field $K(x)$ with*

1. *$[K : \mathbf{Q}] \leq 6$ if the equation is reducible.*
2. *$[K : \mathbf{Q}] \leq 2$ in case $G/Z(G) = F_{36}$.*
3. *$K = \mathbf{Q}$ in all other cases.*

Proof. The method of the proof in section 2 carries over for order 3 equations. Assume that the Riccati equation has at least one algebraic solution. We distinguish three cases.

(1) G is a reducible group. We will restrict ourselves to the worst case and leave the other cases to the reader. Suppose that the vectorspace of solutions of the third order linear differential equation decomposes into a direct sum of three one dimensional G-stable subspaces and suppose that the corresponding Riccati equation has exactly three solutions $u_1, u_2, u_3 \in \mathbf{Q}^{cl}(x)$. Then $\tilde{s}\sigma$ permutes these three solutions for all $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$. We conclude that $u_1, u_2, u_3 \in K(x)$, where $K \subset \mathbf{Q}^{cl}$ is a field such that $Gal(K/\mathbf{Q})$ is isomorphic to a subgroup of S_3 and $[K : \mathbf{Q}] \leq 6$. A rather trivial example is the following:
Let $T^3 + pT + q \in \mathbf{Q}[T]$ denote an irreducible polynomial with Galois group S_3. Then the differential equation $y''' + py' + qy = 0$ has as basis for the solutions $y_i := e^{\alpha_i x}$ where $\alpha_1, \alpha_2, \alpha_3$ are the three zeros of the polynomial $T^3 + pT + q$. The 'only' relation satisfied by y_1, y_2, y_3 over the field $\mathbf{Q}^{cl}(x)$ is $y_1 y_2 y_3 = 1$. The differential Galois group is therefore a maximal torus in $SL(3, \mathbf{Q}^{cl})$ and there are precisely three solutions of the Riccati equation, namely $\alpha_1, \alpha_2, \alpha_3$. This shows that $K = \mathbf{Q}(\alpha_1, \alpha_2)$ is the smallest possible field such that the rational solutions of the Riccati equation are in $K(x)$.

(2) G is an imprimitive group. In this case there are three solutions of the Riccati equation which are cubic over $\mathbf{Q}^{cl}(x)$ and there are no algebraic solutions of lower degree. These three solutions have the same minimum polynomial over $\mathbf{Q}^{cl}(x)$. Hence the coefficients of the minimum polynomial are fixed under $\tilde{s}\sigma$ for all $\sigma \in Gal(\mathbf{Q}^{cl}/\mathbf{Q})$ and therefore the coefficients of the minimum polynomial are in $\mathbf{Q}(x)$.

(3) G is a finite primitive group. Let n be the minimal degree of an algebraic solution of the Ricatti equation. Define the two sets

$$\mathcal{U} = \{u \mid u \text{ is an algebraic solution of the Riccati equation of degree } n\}$$

and

$$\mathcal{H} = \{H \subset G \mid H \text{ is reducible and } [G:H] = n\}.$$

The order of the first set is a multiple of n. Consider the map from $\mathcal{U}$ to $\mathcal{H}$ given by $u \mapsto H_u$ where $H_u := \{\sigma \in G \mid \sigma(u) = u\}$. Clearly H_u is a reducible group and $[G : H_u] = [\mathbf{Q}^{cl}(x,u) : \mathbf{Q}^{cl}(x)] = n$. Conversely one can verify for each primitive group separately that any $H \in \mathcal{H}$ is a non-commutative subgroup of G and therefore fixes only one line in the space of solutions of the third order linear differential equation. Hence H fixes precisely one $u \in \mathcal{U}$. Then one has to count the number of elements in $\mathcal{H}$. Using [SU92], one counts that the order of $\mathcal{H}$ is n except for the case F_{36}. In that case $n = 6$ and the order of $\mathcal{H}$ (and hence of $\mathcal{U}$) is 12. It follows that only in this last case one can possibly have a field K which is a quadratic extension of $\mathbf{Q}$.

Remark. In theorem (5.3) we have for simplicity given a formulation with $\mathbf{Q}$ as field of constants. The remarks of section 3 for the order 2 equations apply also to order 3 equations.

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